HOMEWORK 4

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1. For $z, w \in \mathbb{D}$ define the pseudo-hyperbolic distance to be:

$$\rho(z,w) = \left| \frac{z-w}{1-\bar{w}z} \right|.$$

Show that ρ is a metric that is invariant under automorphisms of $\mathbb{D}$, and that it satisfies the following strenghtening of the triangle inequality:

$$\rho(a,b) \leq \frac{\rho(a,c)+\rho(c,b)}{1+\rho(a,c)\rho(c,b)},$$

for all $a, b, c \in \mathbb{D}$.

Hint: Use the invariance to reduce the case when $c = 0$.

2. Suppose f and g are analytic in $\mathbb{C}$ and $|f(z)| \leq |g(z)|$ for all z. Prove there exist a constant c so that $f(z) = cg(z)$ for all z.

Hint: study the zeros of g.

3. Prove that if f is non-constant and analytic on all of $\mathbb{C}$ then $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Hint: Use the fact that $1/(z-a)$ is analytic for $z \neq a$.

4. Let f be analytic in $\mathbb{D}$ and suppose $|f(z)| < 1$ on $\mathbb{D}$. Let $a = f(0)$. Show that f does not vanish in $\{z : |z| < |a|\}$.

5. Suppose f is analytic in $\mathbb{D}$ and $|f(z)| \leq 1$ in $\mathbb{D}$ and $f(0) = 1/2$. Prove that $|f(1/3)| \geq 1/5$.

Hint: use the invariant form of Schwarz's Lemma.

6. (a) Let $\phi \in \mathrm{Aut}(\mathbb{D})$. Show that

$$\frac{|\phi'(z)|}{1-|\phi(z)|^2} = \frac{1}{1-|z|^2}. \qquad \forall z \in \mathbb{D}.$$

(b) Given a piecewise smooth path $\gamma : [a,b] \to \mathbb{D}$ its hyperbolic length is

$$\ell_h(\gamma) = \int_\gamma \frac{2|dz|}{1-|z|^2} = \int_a^b \frac{2|\gamma'(t)|}{1-|\gamma(t)|^2} dt.$$

Compute the hyperbolic length of the segment $[0, r]$ for $0 < r < 1$.

7. (a) Show that a "circle" C is orthogonal to $\partial\mathbb{D}$ if and only if either C is a line through 0 or $C = \{z : |z-a| = r\}$ with $|a| > 1$ and $r = \sqrt{|a|^2-1}$.

(b) If $z \neq w \in \mathbb{D}$, show there exists a unique $\phi \in \mathrm{Aut}(\mathbb{D})$ such that $\phi(z) = 0$ and $0 < \phi(w) < 1$. Express ϕ in terms of z and w.

(c) If $z \neq w \in \mathbb{D}$, show there is a unique circle C through z and w which is orthogonal to $\partial\mathbb{D}$.

8. The Cayley transform is the map

$$\phi(z) = \frac{1+z}{1-z}.$$

Show that ϕ is a rigid motion of the sphere (an isometry in the chordal distance); find the fixed points; deduce that ϕ sends circle or lines to circles or lines; show that the image of $\mathbb{D}$ under ϕ is the right half-plane $\mathbb{H} = \{\operatorname{Re} z > 0\}$; compute the inverse map. Transfer the pseudo-hyperbolic metric of $\mathbb{D}$ to $\mathbb{H}$. What happens to circles perpendicular to $\partial\mathbb{D}$ under ϕ?

9*. Suppose f is bounded by M and analytic in $\{z : \operatorname{Re} z > 0\}$, and $\limsup_{z \to iy} |f(z)| \leq m < M$ for all iy on the imaginary axis. Prove $|f(z)| \leq m$ on the right half-plane.

Hint: study the auxiliary functions $g_k(z) = f(z)/(1 + z/k)$.

10*. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence 1 and suppose $a_n \geq 0$ for all n. Prove that $z = 1$ is a singular point of f. That is, there is no function g analytic in a neighborhood U of $z = 1$ such that $f = g$ on $U \cap \mathbb{D}$.

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